LCR circuit

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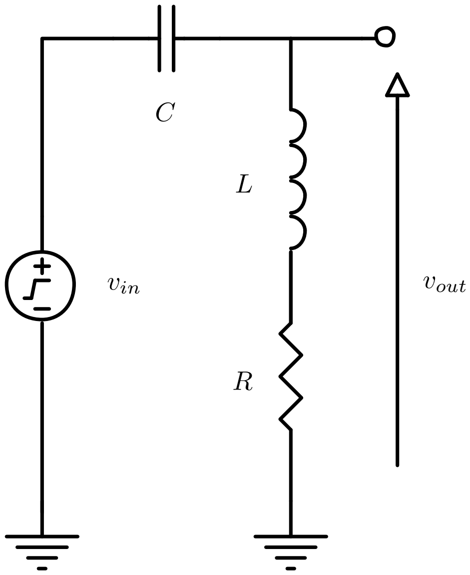


Figure 1: LCR circuit

1 Analysis in the time domain

When V_{in} is a step function given by $V \cdot u(t)$ where V is the magnitude of the step and $u(t) = 1$ for $t \geq 0$ and zero otherwise. The voltage around the loop is given by:

$$Vu(t) = v_c(t) + \frac{di(t)}{dt}L + Ri(t) \tag{1}$$

Substituting $i(t) = \frac{dv_c(t)}{dt}$ into equation 1:

$$Vu(t) = v_c(t) + \frac{d^2v_c(t)}{dt^2}LC + R\frac{dv_c(t)}{dt}C \tag{2}$$

$$\frac{d^2v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_c(t) = \frac{Vu(t)}{LC} \tag{3}$$

Let:

$$v_c(t) = v_f(t) + v_n(t) \tag{4}$$

substituting equation 4 into equation 3:

$$\left[\frac{d^2v_n(t)}{dt^2} + \frac{R}{L}\frac{dv_n(t)}{dt} + \frac{1}{LC}v_n(t)\right] + \left[\frac{d^2v_f(t)}{dt^2} + \frac{R}{L}\frac{dv_f(t)}{dt} + \frac{1}{LC}v_f(t)\right] = 0 + \frac{Vu(t)}{LC} \tag{5}$$

where $v_n(t)$ is the natural response and $v_f(t)$ is the forcing function.

For $t > 0s$ $u(t) = 1$:

$$\left[\frac{d^2v_n(t)}{dt^2} + \frac{R}{L}\frac{dv_n(t)}{dt} + \frac{1}{LC}v_n(t)\right] = 0 \tag{6}$$

$$\left[\frac{d^2v_f(t)}{dt^2} + \frac{R}{L}\frac{dv_f(t)}{dt} + \frac{1}{LC}v_f(t)\right] = \frac{V}{LC} \tag{7}$$

Solve for $v_f(t)$: Since $\frac{V}{LC}$ is a polynomial of degree 0, the solution $v_f(t)$ must be a constant such that:

$$v_f(t) = K \tag{8}$$

$$\frac{dv_f(t)}{dt} = 0 \tag{9}$$

$$\frac{d^2v_f(t)}{dt} = 0 \tag{10}$$

Substituting into equation 7:

$$\frac{1}{LC}K = \frac{V}{LC} \tag{11}$$
$$K = V \tag{12}$$
$$v_f = V \tag{13}$$
$$\tag{14}$$

Solve for $v_n(t)$:

$$\frac{R}{L} = 2\alpha \tag{15}$$
$$\frac{1}{LC} = \omega_n^2 \tag{16}$$
$$v_n(t) = Ae^{st} \tag{17}$$

Substituting into equation 6 gives:

$$\frac{d^2Ae^{st}}{dt^2} + 2\alpha\frac{dAe^{st}}{dt} + \omega_n^2 Ae^{st} = 0 \tag{18}$$
$$s^2Ae^{st} + 2\alpha Ae^{st} + \omega_2^2 Ae^{st} = 0 \tag{19}$$
$$s^2 + 2\alpha s + \omega_n^2 = 0 \tag{20}$$

$$s = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega_n^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} \tag{21}$$

Therefore $v_n(t)$ has two solutions Ae^{s_1t} and Ae^{s_2t} where s_1 and s_2 are given by:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_n^2} \tag{22}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_n^2} \tag{23}$$

The general solution is then given by[3]:

$$v_n(t) = A_1e^{s_1t} + A_2e^{s_2t} \tag{24}$$

Given:

L	R	C	V
0.5	1kΩ	100nF	1V

$$\alpha = \frac{R}{2L} = 1000 \tag{25}$$
$$\omega_n = \frac{1}{\sqrt{LC}} \approx 4472 \tag{26}$$

$$s_1 = -1000 - 4359j \tag{27}$$
$$s_2 = -1000 + 4359j \tag{28}$$
$$\tag{29}$$

$$v_n(t) = A_1 e^{(-1000-4359j)t} + A_2 e^{(-1000-4359j)t} \tag{30}$$

Thus by Euler's formula[2]:

$$e^{j\phi} = \cos\phi + j\sin\phi \tag{31}$$

$$v_n(t) = e^{-1000}\big[\big(A_1\cos(-4359t) + j\sin(-4359t)\big) + A_2\big(\cos(4359t) + j\sin(-4359t)\big)\big]$$
$$= e^{-1000t}\big[(A_1 + A_2)\cos(4359t) + j(-A_1 + A_2)\sin(4359t)\big] \tag{32}$$

Let $B_1 = A_1 + A_2$ and $B_2 = j(-A_1 + A_2)$

$$v_n(t) = e^{-1000t}\big[B_1\cos(4359t) + B_2\sin(4359t)\big] \tag{33}$$

Solve for B_1 and B_2: From equation 13, $v_f = 1$ for a unit step of magnitude 1V. Therefore substitution of v_f and $v_n(t)$ into equation 4 gives:

$$v_c(t) = 1 + e^{-1000t}\big[B_1\cos(4359t) + B_2\sin(4359t)\big] \tag{34}$$

for $t = 0$ the voltage across the capacitor is zero, $v_c(t) = 0$

$$0 = 1 + B_1\cos(0) + B_2\sin(0) \tag{35}$$
$$B_1 = -1 \tag{36}$$

for $t = 0$, the current in the inductor must be zero, $i(0) = 0$

$$i(t) = \frac{dv_c(t)}{dt}C \tag{37}$$

$$i(0) = 100 \cdot 10^{-9}\big[e^{-1000t}\big(- 4359B_1 \sin(4359t) + 4359B_2 \cos(4359t)\big) - 1000e^{-1000t}\big(B_1 \cos(4359t) + B_2 \sin(4359t)\big)\big] \tag{38}$$

$$0 = 100 \cdot 10^{-9}\big[4359B_2 - 1000B_1\big] \tag{39}$$

substituting B_1 from equation 36 gives

$$B_2 \approx -0.229 \tag{40}$$

For $t > 0$, $v_c(t)$ is given by:

$$v_c(t) = 1 - e^{-1000t}\big[\cos(4359t) + 0.229 \sin(4359t)\big] \tag{41}$$

v_{out} is given by:

$$v_{out} = V_{in} - v_c(t) \tag{42}$$

$$v_{out} = Vu(t) - v_c(t) \tag{43}$$

$$\tag{44}$$

For $t > 0$, v_{out} is given by:

$$\boxed{v_{out} = e^{-1000t}\big[\cos(4359t) + 0.229 \sin(4359t)\big]} \tag{45}$$

2 Analysis in the frequency domain

$$H(s) = \frac{sL + R}{\frac{1}{sC} + sL + R} \tag{46}$$

$$= \frac{s^2CL + sRC}{1 + s^2CL + sRC} \tag{47}$$

$$= \frac{sCL\left(s + \frac{R}{L}\right)}{s^2CL + sRC + 1} \tag{48}$$

$$= \frac{s\left(s + \frac{R}{L}\right)}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{49}$$

Let:

$$\frac{R}{L} = 2\alpha \tag{50}$$

$$\frac{1}{LC} = \omega_n^2 \tag{51}$$

$$H(s) = \frac{s\left(s + 2\alpha\right)}{s^2 + 2\alpha s + \omega_n^2} \tag{52}$$

$$= \frac{s\left(s + 2\alpha\right)}{\left(s + \alpha + j\sqrt{\omega_n^2 - \alpha^2}\right)\left(s + \alpha - j\sqrt{\omega_n^2 - \alpha^2}\right)} \tag{53}$$

If $\omega_n^2 > \alpha^2$, then the circuit has complex poles(as in the example presented) and is said to be underdamped[2].

The input v_{in} is a unit step, $u(t)$. The output v_{out} is then given by the convolution of the impulse response h(t) and the unit step function $u(t)$ Therefore the output is given by multiplication in the frequency domain $H(s)U(s)$, where $U(s) = \frac{1}{s}$ is given by the Laplace Transform[1] shown in equation 54.

$$u(t) \leftrightarrow \frac{1}{s} \tag{54}$$

$$H(s)U(s) = \frac{\left(s + 2\alpha\right)}{\left(s + \alpha + j\sqrt{\omega_n^2 - \alpha^2}\right)\left(s + \alpha - j\sqrt{\omega_n^2 - \alpha^2}\right)} \tag{55}$$

We now express $H(s)U(s)$ as the sum of partial fractions[1]:

$$\frac{\left(s + 2\alpha\right)}{\left(s + \alpha + j\sqrt{\omega_n^2 - \alpha^2}\right)\left(s + \alpha - j\sqrt{\omega_n^2 - \alpha^2}\right)} = \frac{A_1}{s + \alpha + j\sqrt{\omega_n^2 - \alpha^2}} + \frac{A_2}{s + \alpha - j\sqrt{\omega_n^2 - \alpha^2}} \tag{56}$$

Solving for A_1

$$\frac{\left(s+2\alpha\right)}{s+\alpha-j\sqrt{\omega_n^2-\alpha^2}} = A_1 + \frac{A_2\left(s+\alpha+j\sqrt{\omega_n^2-\alpha^2}\right)}{s+\alpha-j\sqrt{\omega_n^2-\alpha^2}} \tag{57}$$

$$A_1 = \frac{1}{2} + j\frac{\alpha}{2\sqrt{\omega_n^2-\alpha^2}}\Bigg|_{s=-\alpha-j\sqrt{\omega_n^2-\alpha^2}} \tag{58}$$

Solving for A_2

$$\frac{\left(s+2\alpha\right)}{s+\alpha+j\sqrt{\omega_n^2-\alpha^2}} = \frac{A_1\left(s+\alpha-j\sqrt{\omega_n^2-\alpha^2}\right)}{s+\alpha+j\sqrt{\omega_n^2-\alpha^2}} + A_2 \tag{59}$$

$$A_2 = \frac{1}{2} - j\frac{\alpha}{2\sqrt{\omega_n^2-\alpha^2}}\Bigg|_{s=-\alpha+j\sqrt{\omega_n^2-\alpha^2}} \tag{60}$$

H(s)U(s) then becomes:

$$H(s)U(s) = \frac{\frac{1}{2}+j\frac{\alpha}{2\sqrt{\omega_n^2-\alpha^2}}}{s+\alpha+j\sqrt{\omega_n^2-\alpha^2}} + \frac{\frac{1}{2}-j\frac{\alpha}{2\sqrt{\omega_n^2-\alpha^2}}}{s+\alpha-j\sqrt{\omega_n^2-\alpha^2}} \tag{61}$$

Transformation to the time domain is done with the Laplace Transform pair given in equation 62.

$$2e^{-\alpha t}\big(a\cos(\beta t) - b\sin(\beta t)\big) \leftrightarrow \frac{a+jb}{s+\alpha-j\beta} + \frac{a-jb}{s+\alpha+j\beta} \tag{62}$$

Therefore $h(t) * u(t)$ is given by:

$$h(t) * u(t) = e^{-\alpha t}\Big(\cos(\sqrt{\omega_n^2-\alpha^2}t) + \frac{\alpha}{\sqrt{\omega_n^2-\alpha^2}}\sin(\sqrt{\omega_n^2-\alpha^2}t)\Big) \tag{63}$$

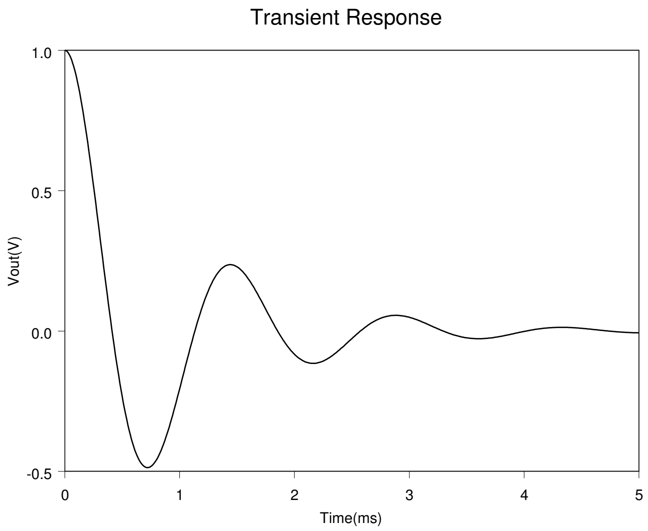
Given:

L	R	C	V
0.5	1kΩ	100nF	1V

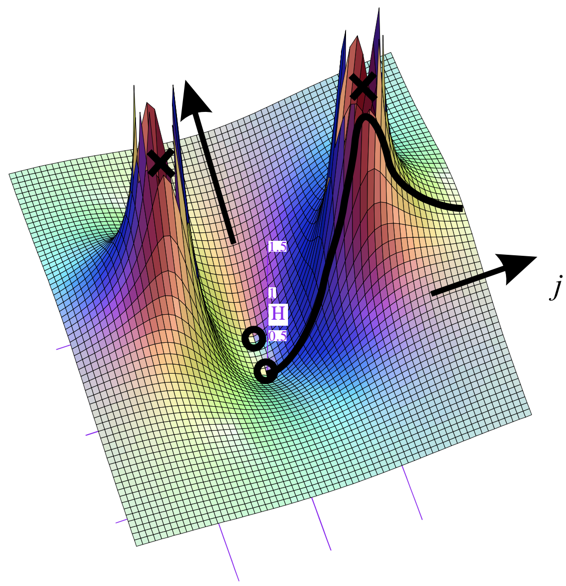
$$\alpha = \frac{R}{2L} = 1000 \tag{64}$$

$$\omega_n = \frac{1}{\sqrt{LC}} \approx 4472 \tag{65}$$

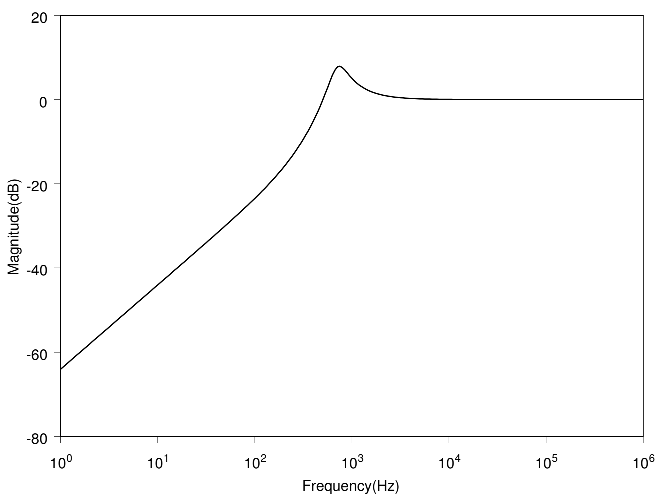
$$\boxed{h(t) * u(t) = e^{-1000t}\big(\cos(4359t) + 0.229\sin(4359t)\big)} \tag{66}$$



Transient Response of LCR circuit



LCR circuit in the S-domain



Frequency Response of LCR circuit

References

[1] Herman J. Blinchikoff and Anatol I. Zverev. *Filtering in the Time and Frequency Domains*. John Wiley and Sons, New York, 1976.

[2] Leonard S. Bobrow. *Fundamentals of Electrical Engineering*. Oxford Univeristy Press, New York, 1996.

[3] James Stewart. *Calculus*. Brooks/Cole Publishing company, CA, 1995.