Skin Effect

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The skin effect is the phenomenon where alternating currents tend to flow near the surface of a conductor. The skin depth is the distance an EM wave incident on a conductor will penetrate before attenuating by $\frac{1}{e}$ where e is the natural number. The following sections derive the skin effect equations from Maxwell's equations.

1 Derive wave equation

Maxwell Equations are[1]:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 J_m \tag{4}$$

Multiplying both sides of equation 3 by ∇ gives:

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} \tag{5}$$

Given[1] $\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - E(\nabla \cdot \nabla)$:

$$\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \frac{\partial(\nabla \times \mathbf{B})}{\partial t} \tag{6}$$

Substitution of equation 4 into equation 6

[1] see appendix for proof

$$\nabla^2\mathbf{E} - \nabla(\nabla\cdot\mathbf{E}) = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 J_m\right) \tag{7}$$

Noting that $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$:

$$\nabla^2\mathbf{E} - \nabla(\nabla\cdot\mathbf{E}) = \frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} + \mu_0 J_m\right) \tag{8}$$

$$\nabla^2\mathbf{E} - \nabla(\nabla\cdot\mathbf{E}) = \mu_0\epsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0\frac{\partial J_m}{\partial t} \tag{9}$$

substituting equation 1:

$$\nabla^2\mathbf{E} - \nabla\left(\frac{\rho_t}{\epsilon_0}\right) = \mu_0\epsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0\frac{\partial J_m}{\partial t} \tag{10}$$

$$\nabla^2\mathbf{E} - \mu_0\epsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0\frac{\partial J_m}{\partial t} = \nabla\left(\frac{\rho_t}{\epsilon_0}\right) \tag{11}$$

Assuming the transmission line does not contain excess charge, the total charge density(ρ_t) in the conductor is zero.

$$\nabla\left(\frac{\rho_t}{\epsilon_0}\right) = 0 \tag{12}$$

The current density J_m is given by:

$$J_m = \sigma E \tag{13}$$

where σ is the metal conductivity.

Substituting equation 12 and equation 13 into equation 11:

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} - \sigma\mu\frac{\partial \mathbf{E}}{\partial t} = 0 \tag{14}$$

2 Define propagating wave

Assume a TEM mode sinusoidal wave is propagating down a transmission line. The conductor has 3 dimensions x, y and z where z is the direction of propagation.

The Electric field may be described by a wave equation of the form:

$$E = E_0\cos(\omega t) \tag{15}$$

Where E_0 is the magnitude of the electric field, ω the angular frequency and t the time

for any position z along the transmission line:

$$E = E_0 \cos\left[\omega(t - \frac{z}{\mu})\right] \tag{16}$$

$$E = E_0 \cos\left[\omega t - \frac{\omega}{\mu} z\right] \tag{17}$$

where μ is the phase velocity.

We then define k as the circular wave number:

$$k = \frac{\omega}{\mu} \tag{18}$$

which gives:

$$E = E_0 \cos(\omega t - kz) \tag{19}$$

The wave may propagate in one of three directions, $\mathbf{i}$(parallel to x axis), $\mathbf{j}$(parallel to y axis) or $\mathbf{k}$(parallel to z axis). In TEM mode, the electric field $\mathbf{E}$ is perpendicular to the magnetic field $\mathbf{H}$ and the vector product $\mathbf{E} \times \mathbf{H}$ points in the direction of propagation. The electric field $\mathbf{E}$ from equation 19 is assigned the direction $\mathbf{i} + \mathbf{j}$, where the electric field $\mathbf{E_j}$ is perpendicular to $\mathbf{E_i}$, such that:

$$\mathbf{E} = E_0 \cos(\omega t - kz)\mathbf{i} + E_0 \sin(\omega t - kz)\mathbf{j} \tag{20}$$

In rectangular coordinates this gives:

$$\mathbf{E} = \big(E_0 \cos(\omega t - kz) + jE_0 \sin(\omega t - kz)\big)\mathbf{i} \tag{21}$$

Using Euler's formula $e^{j\phi} = \cos\phi + j\sin\phi$:

$$\mathbf{E} = E_0 e^{j(\omega t - kz)}\mathbf{i} \tag{22}$$

3 Solve Wave Equation

Solve equation 14 using a complex wave train[2]. Substituting equation 22 into equation 14:

$$\nabla^2\big(E_0 e^{j(\omega t - kz)}\mathbf{i}\big) - \mu\epsilon\frac{\partial^2}{\partial t^2}\big(E_0 e^{j(\omega t - kz)}\mathbf{i}\big) - \sigma\mu\frac{\partial}{\partial t}\big(E_0 e^{j(\omega t - kz)}\mathbf{i}\big) = 0$$

$$-k^2 E_0 e^{j(\omega t - kz)}\mathbf{i} + \mu\epsilon\omega^2 E_0 e^{j(\omega t - kz)}\mathbf{i} - j\omega\sigma\mu E_0 e^{j(\omega t - kz)}\mathbf{i} = 0$$

$$-k^2 + \omega^2\mu\epsilon - j\omega\sigma\mu = 0 \tag{23}$$

$$k^2 = \omega^2\epsilon\mu - j\omega\sigma\mu \tag{24}$$

For good conductors, $j\omega\sigma\mu >> \omega^2\epsilon\mu$ therefore the wave number k reduces to:

$$
\begin{aligned}
k &= \sqrt{-j\omega\sigma\mu} && (25)\\
&= \sqrt{\frac{\omega\sigma\mu}{2}} - j\sqrt{\frac{\omega\sigma\mu}{2}} && (26)\\
&= \frac{1}{\delta} - j\frac{1}{\delta} && (27)
\end{aligned}
$$

Where δ is the skin depth given by:

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu}} \tag{28}$$

such that:

$$
\begin{aligned}
\mathbf{E} &= E_0 e^{j(\omega t - kz)} && (29)\\
&= E_0 e^{j(\omega t - z/\delta + jz/\delta)} && (30)\\
&= E_0 e^{j(\omega t - z/\delta) - z/\delta} && (31)\\
&= E_0 e^{-z/\delta} e^{j(\omega t - z/\delta)} && (32)\\
&= E_0 e^{-z/\delta}\big(\cos(\omega t - z/\delta) + j\sin(\omega t - z/\delta)\big) && (33)\\
&= E_0 e^{-z/\delta}\cos(\omega t - z/\delta) + jE_0 e^{-z/\delta}\sin(\omega t - z/\delta) && (34)
\end{aligned}
$$

The imaginary part of $\mathbf{E}$ indicates the wave is circularly polarized[3]. It then follows:

$$\mathbf{E} = E_0 e^{-z/\delta}\cos(\omega t - z/\delta)\mathbf{i} + E_0 e^{-z/\delta}\sin(\omega t - z/\delta)\mathbf{j} \tag{35}$$

Assuming the propagation of the magnetic field is due primarily to the movement of charge, the transmission line will operate plane-polarized. If the medium is linear(additive and homogenous), then we may apply the principle of superposition. Equation 35 for a plane-polarized transmission line becomes:

$$\mathbf{E} = E_0 e^{-z/\delta}\cos(\omega t - z/\delta)\mathbf{i} \tag{36}$$

A Appendix

Proof for: $\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - E(\nabla \cdot \nabla)$

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{37}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{38}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \tag{39}$$

$$\begin{aligned} \nabla \times \nabla \times \mathbf{F} =& \left(\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right)\mathbf{i} \\ & \left(\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right)\mathbf{j} \\ & \left(\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right)\mathbf{k} \end{aligned} \tag{40}$$

$$\begin{aligned} =& \left(\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right)\mathbf{i} \\ +& \left(\frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial y \partial z} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial z^2}\right)\mathbf{j} \\ +& \left(\frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2}\right)\mathbf{j} \end{aligned} \tag{41}$$

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\left(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}\right) \tag{42}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{43}$$

$$\nabla(\nabla \cdot \mathbf{F}) = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \tag{44}$$

$$\begin{aligned} &= \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}\right)\mathbf{i} \\ &+ \left(\frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}\right)\mathbf{j} \\ &+ \left(\frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial z^2}\right)\mathbf{k} \end{aligned} \tag{45}$$

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{46}$$

$$\mathbf{F}(\nabla \cdot \nabla) = \left(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \tag{47}$$

$$\begin{aligned} &= \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\mathbf{i} \\ &+ \left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}\right)\mathbf{j} \\ &+ \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right)\mathbf{j} \end{aligned} \tag{48}$$

$$\begin{aligned} \nabla(\nabla \cdot F) - F(\nabla \cdot \nabla) &= \left(\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right)\mathbf{i} \\ &+ \left(\frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial y \partial z} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial z^2}\right)\mathbf{j} \\ &+ \left(\frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2}\right)\mathbf{j} \end{aligned} \tag{49}$$

$$= \nabla \times \nabla \times \mathbf{F} \tag{50}$$

References

[1] P. Lorrain and D. R. Corson, *Electromagnetic fields and waves.* San Francisco: W.H. Freeman, 1970.

[2] R. Knobel, *An Introduction to the Mathematical Theory of Waves.* Providence, R.I.: American Mathematical Society: Institute for Advanced Study, 2000.

[3] I. Tolstoy, *Wave Propagation.* New York: McGraw-Hill, 1973.