Fourier Series: Square Wave

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January 6, 2004

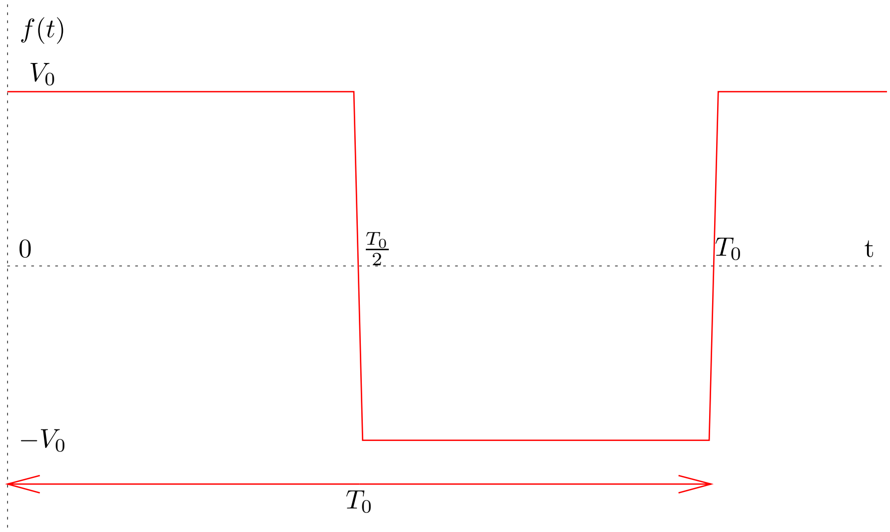

Figure 1: Square wave f(t)

The fourier series of a periodic continuous function is defined by:

$$y(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi mt}{T_0}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{2\pi mt}{T_0}\right)$$

where T_0 is the functions fundamental period and a_0, a_m, b_m are defined as:

$$a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)\,dt$$

$$a_m = \frac{2}{T_0}\int_0^{T_0} f(t)\cos\left(\frac{2\pi mt}{T_0}\right) dt$$

$$b_m = \frac{2}{T_0}\int_0^{T_0} f(t)\sin\left(\frac{2\pi mt}{T_0}\right) dt$$

For the square wave in figure 1; a_0, a_m, b_m are given by:

$$
\begin{aligned}
a_0 &= 0 \\
a_m &= \frac{2}{T_0}\int_0^{\frac{T_0}{2}} V_0 \cos\left(\frac{2\pi mt}{T_0}\right), dt + \frac{2}{T_0}\int_{\frac{T_0}{2}}^{T_0} -V_0\cos\left(\frac{2\pi mt}{T_0}\right), dt \\
&= \frac{2}{T_0}\left[\frac{T_0V_0}{2\pi m}\sin\left(\frac{2\pi mt}{T_0}\right)\right]_0^{\frac{T_0}{2}} + \frac{2}{T_0}\left[-\frac{T_0V_0}{2\pi m}\sin\left(\frac{2\pi mt}{T_0}\right)\right]_{\frac{T_0}{2}}^{T_0} \\
&= \frac{2}{T_0}\left[\frac{T_0V_0}{2\pi m}\sin\left(\frac{2\pi mT_0}{2T_0}\right) - \frac{T_0V_0}{2\pi m}\sin\left(\frac{2\pi m\cdot 0}{T_0}\right)\right] \\
&+ \frac{2}{T_0}\left[-\frac{T_0V_0}{2\pi m}\sin\left(\frac{2\pi mT_0}{T_0}\right) + \frac{T_0V_0}{2\pi m}\sin\left(\frac{2\pi mT_0}{2T_0}\right)\right] \\
&= \frac{2}{T_0}\left[\frac{2T_0V_0}{2\pi m}\sin\left(m\pi\right) - \frac{T_0V_0}{2\pi m}\sin\left(2\pi m\right)\right] \\
&= \frac{2V_0}{m\pi}\sin\left(m\pi\right) - \frac{V_0}{m\pi}\sin\left(2\pi m\right) \\
&= 0 \text{ for all } m \in \mathbf{N}
\end{aligned}
$$

$$
\begin{aligned}
b_m &= \frac{2}{T_0}\int_0^{\frac{T_0}{2}} V_0 \sin\left(\frac{2\pi mt}{T_0}\right), dt + \frac{2}{T_0}\int_{\frac{T_0}{2}}^{T_0} -V_0\sin\left(\frac{2\pi mt}{T_0}\right), dt \\
&= \frac{2}{T_0}\left[-\frac{T_0V_0}{2\pi m}\cos\left(\frac{2\pi mt}{T_0}\right)\right]_0^{\frac{T_0}{2}} + \frac{2}{T_0}\left[-\frac{T_0V_0}{2\pi m}\cos\left(\frac{2\pi mt}{T_0}\right)\right]_{\frac{T_0}{2}}^{T_0} \\
&= \frac{2}{T_0}\left[-\frac{T_0V_0}{2\pi m}\cos\left(\frac{2\pi mT_0}{2T_0}\right) + \frac{T_0V_0}{2\pi m}\cos\left(\frac{2\pi m\cdot 0}{T_0}\right)\right] \\
&+ \frac{2}{T_0}\left[\frac{T_0V_0}{2\pi m}\cos\left(\frac{2\pi mT_0}{T_0}\right) - \frac{T_0V_0}{2\pi m}\cos\left(\frac{2\pi mT_0}{2T_0}\right)\right] \\
&= \frac{2}{T_0}\left[\frac{T_0V_0}{2\pi m}\cos\left(2\pi m\right) - \frac{T_0V_0}{\pi m}\cos\left(\pi m\right) + \frac{T_0V_0}{2\pi m}\right] \\
&= \frac{V_0}{\pi m}\left[1 + \cos\left(2\pi m\right) - 2\cos\left(\pi m\right)\right]
\end{aligned}
$$

The coefficients are given by:

m	1	2	3	4	5	6	7
a_m	0	0	0	0	0	0	0
b_m	$\frac{4V_0}{\pi}$	0	$\frac{4V_0}{3\pi}$	0	$\frac{4V_0}{5\pi}$	0	$\frac{4V_0}{7\pi}$

which gives the fourier series:

$$y(t) = \frac{4V_0}{\pi}\sin\left(\frac{2\pi t}{T_0}\right) + \frac{4V_0}{3\pi}\sin\left(\frac{6\pi t}{T_0}\right) + \frac{4V_0}{5\pi}\sin\left(\frac{10\pi t}{T_0}\right) + \frac{4V_0}{7\pi}\sin\left(\frac{14\pi t}{T_0}\right) + ...$$

Letting $f_0 = \frac{2\pi}{T_0}$ the fourier series becomes:

$$y(t) = \frac{4V_0}{\pi} \sin\left(f_0 t\right) + \frac{4V_0}{3\pi} \sin\left(3 f_0 t\right) + \frac{4V_0}{5\pi} \sin\left(5 f_0 t\right) + \frac{4V_0}{7\pi} \sin\left(7 f_0 t\right) + ...$$

The magnitude spectrum corresponding to the square wave of figure 1 is given in figure 2.

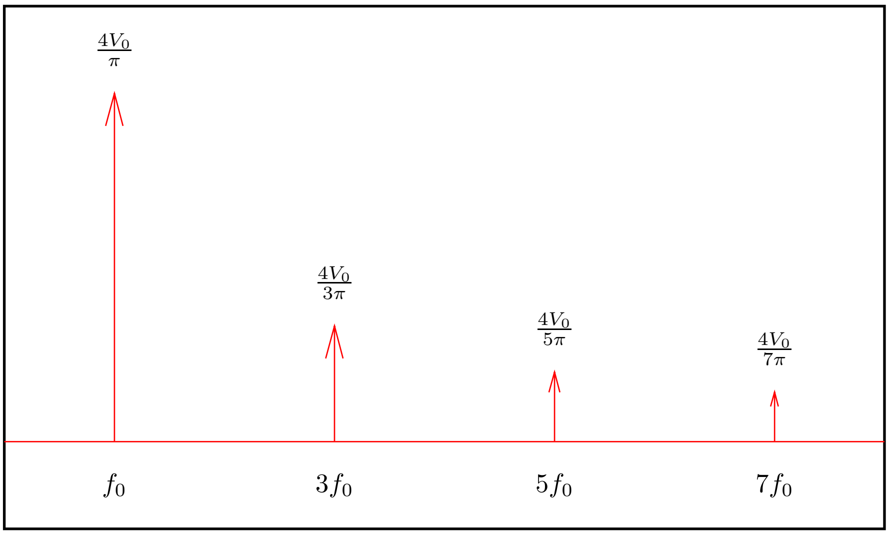

Figure 2: Square wave magnitude spectrum $|F(f)|$