

Vacuum Tubes

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September 21, 2002

Abstract

While the vacuum tube or valve appears to be obsolete, it is still applicable in a number of application. One of these applications is audio amplifiers. It is claimed that valve amplifiers produce 'better' sound than their solid state counterparts. Valves have a number of obvious disadvantages, such as: size, efficiency, lifespan and cost. Thus duplicating the valve sound with a solid state equivalent would have significant benefits. This requires an understanding of the underlying distortion mechanisms.

1 Vacuum Devices

1.1 The Diode

One may start the analysis of vacuum devices with the simplest case, the vacuum diode. This consists of an emitter of electrons, the cathode(K) and a collector of electrons the anode(A). Electrons in the cathode are excited by an external force, such that electrons may exist in free space. If the anode is sufficiently positive, these electrons drift towards the anode, generating current.

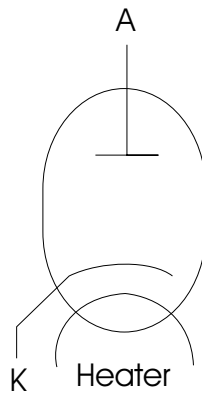


Figure 1: Valve Diode

We shall begin this discussion by assuming there exists free electrons with zero velocity at the cathode available to drift towards the anode.

If one applies a positive external electric field to the tube, between the anode and cathode, the electrons will experience a force in the direction of the anode. Consequently they will accelerate towards the anode where they will be collected.

One may now derive a simple expression for the current density J (amps/m²) in the tube, given by:

$$J = \rho v \tag{1}$$

where v is the drift velocity and ρ is the electron density(Coulombs/m²).

The velocity is given by the energy of the electrons:

$$\frac{1}{2}mv^2 = qV \quad (2)$$

$$v = \sqrt{\frac{2qV}{m}} \quad (3)$$

where q is the charge of an electron and m its mass.

The electron density ρ is given by the solution of Poisson's equations in one dimension:

$$\frac{d^2V}{dx^2} = \frac{\rho}{\epsilon_0} \quad (4)$$

$$\rho = \frac{d^2V}{dx^2} \epsilon_0 \quad (5)$$

where x is the distance in the tube relative to the cathode and ϵ_0 is the permittivity of free space.

Thus substituting into J one gets the differential equation:

$$J = \frac{d^2V}{dx^2} \epsilon_0 \sqrt{\frac{2qV}{m}} \quad (6)$$

Solving for J:

$$\text{Let: } y = \frac{dV}{dx} \quad dx = \frac{dV}{y} \quad (7)$$

$$\frac{dy}{dx} = \frac{J}{\epsilon_0} \left(\frac{2qV}{m} \right)^{-\frac{1}{2}} \quad (8)$$

$$dy = \frac{J}{\epsilon_0} \left(\frac{2q}{m} \right)^{-\frac{1}{2}} V^{-\frac{1}{2}} dx \quad (9)$$

$$dy = \frac{J}{\epsilon_0} \left(\frac{2q}{m} \right)^{-\frac{1}{2}} V^{-\frac{1}{2}} \frac{dV}{y} \quad (10)$$

$$y dy = \frac{J}{\epsilon_0} \left(\frac{2q}{m} \right)^{-\frac{1}{2}} V^{-\frac{1}{2}} dV \quad (11)$$

$$\int y dy = \int \frac{J}{\epsilon_0} \left(\frac{2q}{m} \right)^{-\frac{1}{2}} V^{-\frac{1}{2}} dV \quad (12)$$

$$\frac{1}{2} y^2 = 2 \frac{J}{\epsilon_0} \left(\frac{2q}{m} \right)^{-\frac{1}{2}} V^{\frac{1}{2}} + C \quad (13)$$

$$\left(\frac{dV}{dx} \right)^2 = 4 \frac{J}{\epsilon_0} \left(\frac{2q}{m} \right)^{-\frac{1}{2}} V^{\frac{1}{2}} + C_1 \quad (14)$$

It will be shown later that under normal operating conditions, given the voltage at the cathode(x=0) is zero, $\frac{dV}{dx} = 0$. Therefore the integration constant(C_1) goes to zero.

$$\frac{dV}{dx} = \sqrt{4 \frac{J}{\epsilon_0} \left(\frac{2q}{m} \right)^{-\frac{1}{2}} V^{\frac{1}{2}}} \quad (15)$$

$$\frac{dV}{dx} = 2 \left(\frac{J}{\epsilon_0} \right)^{\frac{1}{2}} \left(\frac{2q}{m} \right)^{-\frac{1}{4}} V^{\frac{1}{4}} \quad (16)$$

$$V^{-\frac{1}{4}} dV = 2 \left(\frac{J}{\epsilon_0} \right)^{\frac{1}{2}} \left(\frac{2q}{m} \right)^{-\frac{1}{4}} dx \quad (17)$$

$$\int V^{-\frac{1}{4}} dV = \int 2 \left(\frac{J}{\epsilon_0} \right)^{\frac{1}{2}} \left(\frac{2q}{m} \right)^{-\frac{1}{4}} dx \quad (18)$$

$$\frac{4}{3} V^{\frac{3}{4}} = 2 \left(\frac{J}{\epsilon_0} \right)^{\frac{1}{2}} \left(\frac{2q}{m} \right)^{-\frac{1}{4}} x + C_2 \quad (19)$$

$$(20)$$

At the cathode(x=0), the voltage goes to zero. Therefore the constant of integration C_2 goes to zero.

Thus:

$$\frac{4}{3}V^{\frac{3}{4}} = 2\left(\frac{J}{\epsilon_0}\right)^{\frac{1}{2}}\left(\frac{2q}{m}\right)^{-\frac{1}{4}}x \quad (21)$$

$$\frac{16}{9}V^{\frac{6}{4}} = 4\frac{J}{\epsilon_0}\left(\frac{2q}{m}\right)^{\frac{1}{2}}x^2 \quad (22)$$

$$J = \frac{4\epsilon_0}{9}\left(\frac{2q}{m}\right)^{\frac{1}{2}}\frac{V^{\frac{3}{2}}}{x^2} \quad (23)$$

Therefore the current I is some function of the tube dimensions and the anode voltage:

$$I = \frac{A}{x^2}\frac{4\epsilon_0}{9}\left(\frac{2q}{m}\right)^{\frac{1}{2}}V^{\frac{3}{2}} \quad (24)$$

where A is the cross sectional area of a parallel anode diode tube.

1.1.1 The Triode

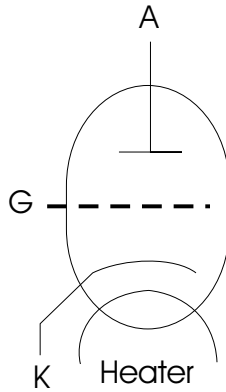


Figure 2: Triode

The triode is a vacuum diode with an additional grid(wire mesh) inserted between the cathode and anode. The grid has large spacing between adjacent wires such that most of the current passes through the grid and is collected by the anode. Placing the grid close to the cathode ensures the grid has significant effect on the emitted current while drawing little current itself.

Under normal operation, the electrons are accelerated towards the higher potential of the anode. If one holds the grid at a negative voltage with respect

to the cathode, this significantly alters the electric field and retards the flow of electrons. This is clearly illustrated in figure 3.

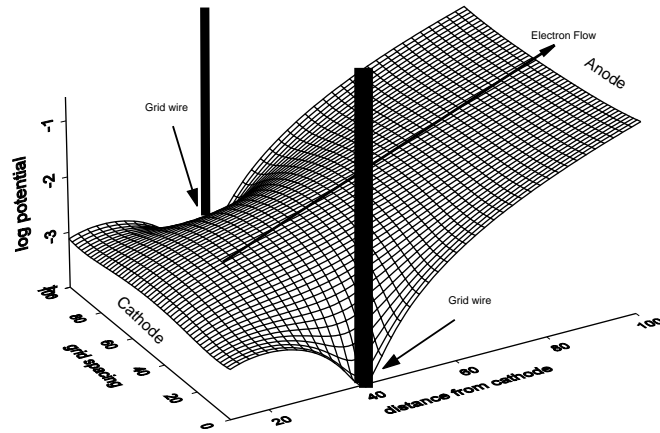


Figure 3: Electric Potential in tube

If one continues to decrease the grid potential, this will further restrict electron flow and eventually the triode will enter cut-off. As the grid is physically closer to the cathode it will exert stronger control on electron flow. To define the relative strength of the grid and the anode an amplification factor μ is defined. This gives the equation [1].

$$i_a = G \left(V_g + \frac{V_a}{\mu} \right)^{3/2} \quad (25)$$

where V_g, V_a are the Grid and Anode voltages respectively. The constant G is called the "perveance". No rigorous theoretical derivation for this equation exists, however it may be verified experientially[1].

1.1.2 Electron emission

We began the discussion of vacuum tubes by assuming there was a supply of electrons with zero velocity ready to drift towards the anode. This section expands on that assumption.

Given two conductors separated by some insulator, electrons don't tend to drift from one to another. In order for electrons to escape from the metal, they must overcome the binding energy of the conductor. The minimum energy required for this is given as the work function of the metal, ϕ_m and is shown in figure 4.

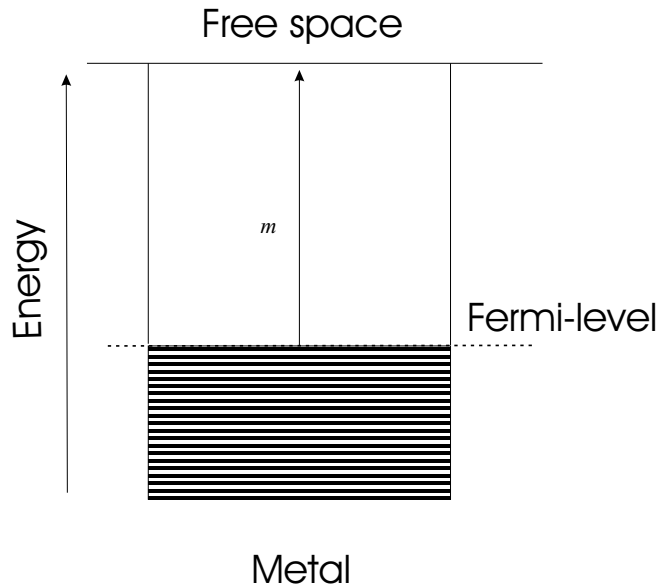


Figure 4: Work function in Metal

There are several method for giving an electrons this energy. A simply method is to increase the temperature of the conductor. The electrons will then gain random amount of energy due to the thermal excitation. The number of electrons with energy greater than the work function is given statistically by the Fermi-Dirac function[3]:

$$P_f(W) = \frac{1}{1 + e^{\frac{W - W_F}{kT}}} \quad (26)$$

$$P_f(W) = \frac{1}{1 + e^{\frac{e\phi_m}{kT}}} \quad (27)$$

The number of these electrons that are made available to drift to the anode is limited by the 'Space-charge'. As electrons escape the metal, they collect at the cathode. This collection of electrons(space-charge) creates an electric field in the opposite direction to the externally applied field. Increasing the space-charge will increase the force towards the cathode and hence deplete available charge.

Depleting the charge will decrease the force towards the cathode hence reducing the number of electrons migrating back to the cathode. Thus at equilibrium, the force due to the space charge must equal the force due to the external field. This is convenient as zero force implies zero velocity by the basic equation 28.

$$F = \frac{1}{2}mv^2 \tag{28}$$

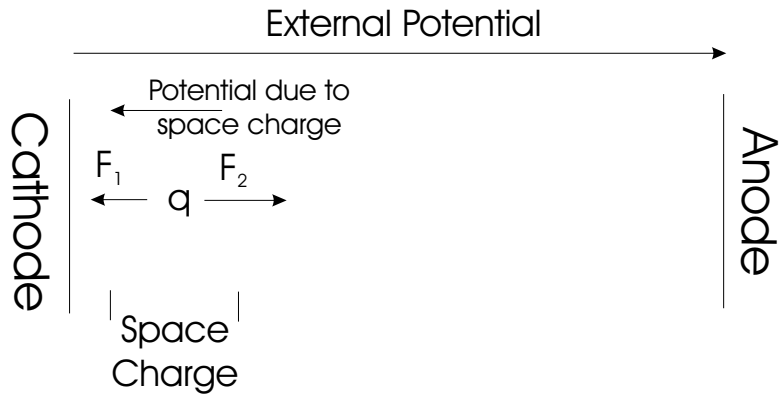


Figure 5: Effect of space charge

This assumes that the electrons arrive with zero initial velocity. Although a majority of electrons will have close to zero velocity, by equation 27. Electrons will exist with large initial velocities. This creates a potential barrier of a fraction of 1eV[1]. Consequently this shifts the space-charge towards the anode. Thus one creates a virtual cathode, shifted towards the anode and decreased in potential.

2 Triode Distortion

Vacuum devices are still applicable in a number of applications. One application which has persisted is audio. It is claimed by a large community of audiophiles that vacuum tubes produce "better" audio. Many claim that vacuum tubes are more linear and produce less "harsh sound". Solid state amplifiers are easily build with distortion that is -80dB for a class AB amplifier and -100dB or less dynamic range for class A amplifiers.

It appears more likely the distortion of a tube amplifier enhances the music. It is widely believed that 2nd order distortion sounds "better" than 3rd order distortion. It may be possible that valve amplifiers mask out the harsh sound of 3rd order distortion with a pleasant 2nd order distortion. For this discussion, we will consider the non-linear distortion of the triode amplifier.

The distortion produced by a tube amplifier depends on the topology. Two common topologies are investigated, the single-ended amplifier and the push-pull amplifier.

2.1 Single-ended Topology

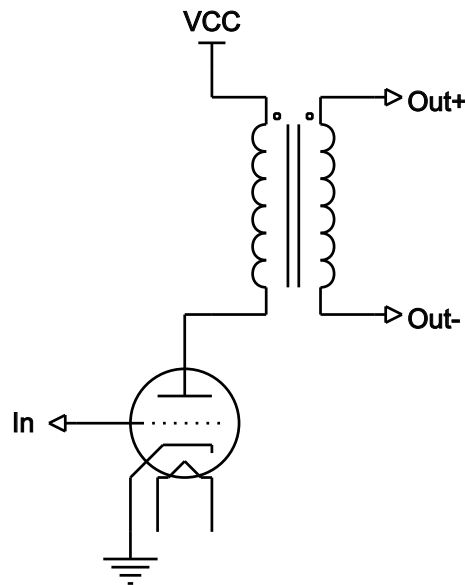


Figure 6: single-ended Transfer Characteristic

The single ended topology is simple to implement and as the transistor is

always on, it does not suffer from cross over distortion. The transfer characteristic is given by the triode equation, 25. As this is not a polynomial, one can not readily discern the distortion products it will produce. The transfer function may be represented with a power series $g(x)$ given below.

$$g(x) = ax + bx^2 + cx^3 + \dots \quad (29)$$

Assuming a constant anode voltage, the effect of distortion for a sinusoidal excitation $x = A_m \cos(\omega t)$ may now be seen.

$$g(x) = A_m a \cos(\omega t) + A_m^2 b \cos^2(\omega t) + A_m^3 c \cos(\omega t) \quad (30)$$

The second order non-linearity of the valve transfer function will produce distortion which is a function of A_m^2 . The third order non-linear distortion a function of A_m^3 , ect. Therefore the THD(total harmonic distortion), which is a ratio of the linear signal to the distortion products will be some function of amplitude. Thus as signal amplitude changes so does the THD and the dominant distortion type(odd/even). It may be possible this type of distortion mechanism sounds harsh.

$$THD = \frac{aA_m}{bA_m^2 + cA_m^3 + \dots} = \frac{a}{bA_m + cA_m^2 + \dots} \quad (31)$$

2.2 Push-pull topology

A push-pull amplifier has the significant advantage of efficiency. The push-pull topology may be seen in figure 7.

One may begin the distortion analysis by assuming the upper valve has the same characteristics as the lower valve. Using equation 25 and assuming the Anode voltage is held constant, this gives the transfer characteristic in figure 8. While this does not represent the circuit non-linearity, it may be used as a figure of merit.

To analyze the distortion this transfer characteristic generates, assume we can represent the upper and lower characteristic by a truncated power series such that:

$$g(x) = a_1x + b_1x^2 + c_1x^3 \quad (32)$$

$$g(-x) = -a_2x + b_2x^2 - c_1x^3 \quad (33)$$

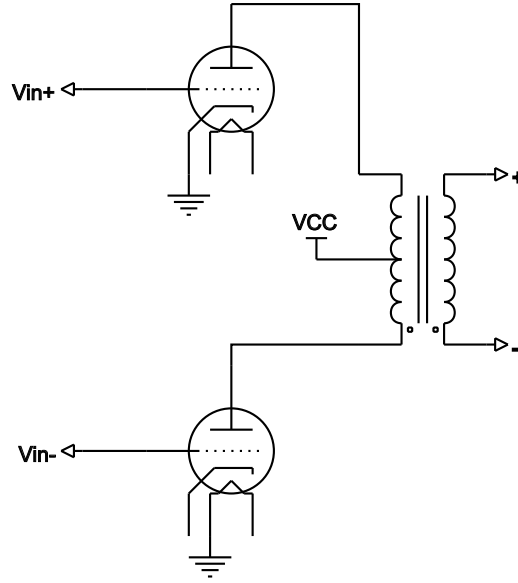


Figure 7: Push Pull Topology

The frequency domain response is then given by the fourier series:

$$f(t) = \frac{A_0}{T} + \frac{2}{T} \sum_{n=1}^{\infty} (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) \quad (34)$$

where A_n is the envelope of the frequency spectrum and $B_n = 0$ for a non-linear memoryless system with excitation $A_m \cos(\omega t)$.

Figure 9 shows the operation of the push-pull amplifier over one period of a sinusoidal excitation. The signal is broken up into different operating regions, one for the upper valve and two for the lower valve.

Thus the envelope of the fourier transform, A_n may be given in terms of these regions:

$$A_n = \int_{-\frac{T}{2}}^{-\frac{T}{4}} g(-x) \cos(\omega_n t) dt + \int_{-\frac{T}{4}}^{\frac{T}{4}} g(x) \cos(\omega_n t) dt + \int_{\frac{T}{4}}^{\frac{T}{2}} g(-x) \cos(\omega_n t) dt \quad (35)$$

where $\omega_n = \frac{2\pi n}{T}$, $\omega = \frac{2\pi}{T}$ and $x = -x = A_m \cos(\omega t)$ as both upper and lower valve are on for positive excitation (however different phase). As one may see in

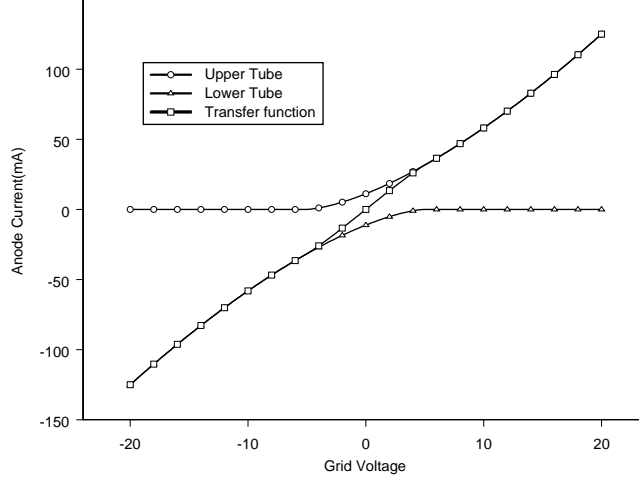


Figure 8: Push Pull Transfer Characteristic

the appendix, one cannot derive a function for a_n as it is not defined for all n . However it may be evaluated for a given n .

We may thus calculate the magnitude of the 2nd order distortion($n=2$).

$$A_2 = A_m a_2 \int_{-\frac{T}{2}}^{-\frac{T}{4}} \cos(\omega t) \cos(\omega_n t) dt - A_m^2 b_2 \int_{-\frac{T}{2}}^{-\frac{T}{4}} \cos^2(\omega t) \cos(\omega_n t) dt \quad (36)$$

$$+ A_m^3 c_2 \int_{-\frac{T}{2}}^{-\frac{T}{4}} \cos^3(\omega t) \cos(\omega_n t) dt + A_m a_1 \int_{-\frac{T}{4}}^{\frac{T}{4}} \cos(\omega t) \cos(\omega_n t) dt \quad (37)$$

$$+ A_m^2 b_1 \int_{-\frac{T}{4}}^{\frac{T}{4}} \cos^2(\omega t) \cos(\omega_n t) dt + A_m^3 c_1 \int_{-\frac{T}{4}}^{\frac{T}{4}} \cos^3(\omega t) \cos(\omega_n t) dt \quad (38)$$

$$+ A_m a_2 \int_{\frac{T}{4}}^{\frac{T}{2}} \cos(\omega t) \cos(\omega_n t) dt - A_m^2 b_2 \int_{\frac{T}{4}}^{\frac{T}{2}} \cos^2(\omega t) \cos(\omega_n t) dt \quad (39)$$

$$+ A_m^3 c_2 \int_{\frac{T}{4}}^{\frac{T}{2}} \cos^3(\omega t) \cos(\omega_n t) dt \quad (40)$$

Substituting from the appendix:

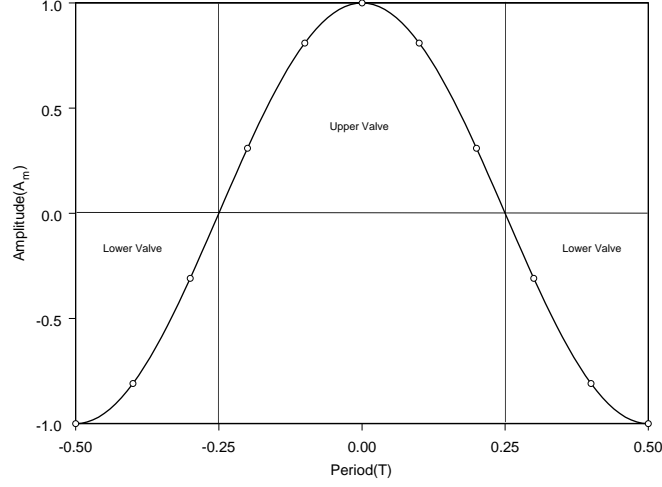


Figure 9: Valve operating region for sinusoidal waveform

$$A_2 = \frac{A_m a_1 T \sin\left(\frac{3\pi}{2}\right)}{6\pi} - \frac{A_m a_1 T \sin\left(\frac{3\pi}{2}\right)}{2\pi} \quad (41)$$

$$- \frac{A_m a_2 T \sin\left(\frac{3\pi}{2}\right)}{6\pi} + \frac{A_m a_2 T \sin\left(\frac{3\pi}{2}\right)}{2\pi} \quad (42)$$

$$+ \frac{A_m^2 b_1 T}{8} - \frac{A_m^2 b_2 T}{8} \quad (43)$$

$$+ \frac{A_m^3 c_1}{5\pi} + \frac{A_m^3 c_1}{5\pi} - \frac{A_m^3 c_2}{5\pi} - \frac{A_m^3 c_2}{5\pi} \quad (44)$$

$$A_2 = A_m (a_1 - a_2) \left(\frac{T}{3\pi}\right) + A_m^2 (b_1 - b_2) \left(\frac{T}{8}\right) + A_m^3 (c_1 - c_2) \left(\frac{2}{5\pi}\right) \quad (45)$$

Therefore the second order distortion is a function of the mismatch between the upper and lower valve transfer characteristics[2]. Although the upper and lower valve may be of the same type, their physical dimensions and position of the virtual cathode may vary greatly with process and time. This effect would also be exacerbated for minimalist designs which encompass little or no feedback.

Given that the linear transfer characteristic is dominant, as one would expect

in an audio amplifier, the second order distortion would be a function of $A_m(a_1 - a_2)$, which is a linear function of signal amplitude. Hence the THD produced by this would take the form:

$$THD \propto \frac{aA_m}{(a_1 - a_2)A_m} = \frac{a}{a_1 - a_2} \quad (46)$$

where a is the linear transfer characteristic and $(a_1 - a_2)$ is the mismatch of this characteristic between the upper and lower valve. Thus the distortion would remain constant for changes in signal amplitude and 'mask' other distortion generating mechanisms. It may be possible that second order distortion that is constant with signal amplitude is less 'harsh' and hence more pleasing.

3 Conclusion

While vacuum tube devices exhibit strong second order distortion in their transfer characteristic, this would be cancelled out in a push-pull amplifier. It may be possible that significant second order distortion results from mismatch between the valves, which results in a constant THD over signal amplitude. It is thought the resulting distortion 'masks' out the other distortion mechanisms, producing less 'harsh' sound.

4 Appendix

Expansion of integrals: one can not easily derive the equation for the envelope of the distortion products. This is due to undefined terms in the evaluation of the integrals. For instance when $n = 1$, $(\omega - \omega_n)$ goes to zero and $\int \cos(\omega t) \cos(\omega_n t) dt$ becomes undefined. This may be overcome by evaluating each point at n .

$$\int \cos(\omega t) \cos(\omega_n t) dt \quad (47)$$

$$= \frac{1}{2} \int \cos((\omega + \omega_n)t) dt + \frac{1}{2} \int \cos((\omega - \omega_n)t) dt \quad (48)$$

$$= \frac{\sin((\omega + \omega_n)t)}{2(\omega + \omega_n)} + \frac{\sin((\omega - \omega_n)t)}{2(\omega - \omega_n)} + C \quad (49)$$

$$\int \cos^2(\omega t) \cos(\omega_n t) dt \quad (50)$$

$$= \frac{1}{2} \int (\cos(2\omega t) + 1) \cos(\omega_n t) dt \quad (51)$$

$$= \frac{1}{2} \int \cos(2\omega t) \cos(\omega_n t) dt + \frac{1}{2} \int \cos(\omega_n t) dt \quad (52)$$

$$= \frac{\sin((2\omega + \omega_n)t)}{4(2\omega + \omega_n)} + \frac{\sin((2\omega - \omega_n)t)}{4(2\omega - \omega_n)} + \frac{\sin(\omega_n t)}{2\omega_n} + C \quad (53)$$

$$\int \cos^3(\omega t) \cos(\omega_n t) dt \quad (54)$$

$$= \int \cos^2(\omega t) \cos(\omega t) \cos(\omega_n t) dt \quad (55)$$

$$= \frac{1}{2} \int (\cos(2\omega t) + 1) \cos(\omega t) \cos(\omega_n t) dt \quad (56)$$

$$= \frac{1}{2} \int \cos(2\omega t) \cos(\omega t) \cos(\omega_n t) dt + \frac{1}{2} \int \cos(\omega t) \cos(\omega_n t) dt \quad (57)$$

$$= \frac{1}{4} \int (\cos(2\omega t) \cos((\omega + \omega_n)t) dt \quad (58)$$

$$+ \frac{1}{4} \int (\cos(2\omega t) \cos((\omega - \omega_n)t) dt \quad (59)$$

$$+ \frac{1}{4} \int \cos((\omega + \omega_n)t) dt + \frac{1}{4} \int \cos((\omega - \omega_n)t) dt \quad (60)$$

$$= \frac{1}{8} \int (\cos(3\omega + \omega_n)t) dt + \frac{1}{8} \int (\cos(\omega - \omega_n)t) dt \quad (61)$$

$$+ \frac{1}{8} \int (\cos(3\omega - \omega_n)t) dt + \frac{1}{8} \int (\cos(\omega + \omega_n)t) dt \quad (62)$$

$$+ \frac{1}{4} \int \cos((\omega + \omega_n)t) dt + \frac{1}{4} \int \cos((\omega - \omega_n)t) dt \quad (63)$$

$$= \frac{1}{8} \int \cos((3\omega + \omega_n)t) dt + \frac{1}{8} \int \cos((3\omega - \omega_n)t) dt \quad (64)$$

$$+ \frac{3}{8} \int \cos((\omega + \omega_n)t) dt + \frac{3}{8} \int \cos((\omega - \omega_n)t) dt \quad (65)$$

$$= \frac{\sin((3\omega + \omega_n)t)}{8(3\omega + \omega_n)} + \frac{\sin((3\omega - \omega_n)t)}{8(3\omega - \omega_n)} \quad (66)$$

$$+ \frac{3 \sin((\omega + \omega_n)t)}{8(\omega + \omega_n)} + \frac{3 \sin((\omega - \omega_n)t)}{8(\omega - \omega_n)} + C \quad (67)$$

For $n = 2$

$$\omega = \frac{2\pi}{T} \quad (68)$$

$$\omega_n = \frac{2\pi n}{T} = \frac{4\pi}{T} = 2\omega \quad (69)$$

$$\int \cos(\omega t) \cos(\omega_n t) dt \quad (70)$$

$$= \int \frac{1}{2} \cos(\omega + \omega_n) + \frac{1}{2} \cos(\omega - \omega_n) dt \quad (71)$$

$$= \int \frac{1}{2} \cos\left(\frac{6\pi}{T}t\right) + \frac{1}{2} \cos\left(\frac{2\pi}{T}t\right) dt \quad (72)$$

$$= \frac{T}{12\pi} \sin\left(\frac{6\pi}{T}t\right) + \frac{T}{4\pi} \sin\left(\frac{2\pi}{T}t\right) + C \quad (73)$$

$$\int \cos^2(\omega t) \cos(\omega_n t) dt \quad (74)$$

$$= \int \frac{1}{2} (\cos(2\omega t) + 1) \cos(\omega_n t) dt \quad (75)$$

$$= \int \frac{1}{2} \cos^2(\omega_n t) + \frac{1}{2} \cos(\omega_n t) dt \quad (76)$$

$$= \int \frac{1}{4} \cos(2\omega_n t) + \frac{1}{2} \cos(\omega_n t) + \frac{1}{4} dt \quad (77)$$

$$= \int \frac{1}{4} \cos\left(\frac{8\pi}{T}t\right) + \frac{1}{2} \cos\left(\frac{4\pi}{T}t\right) + \frac{1}{4} dt \quad (78)$$

$$= \frac{T}{32\pi} \sin\left(\frac{8\pi}{T}t\right) + \frac{T}{8\pi} \sin\left(\frac{4\pi}{T}t\right) + \frac{t}{4} + C \quad (79)$$

$$\int \cos^3(\omega t) \cos(\omega_n t) dt \quad (80)$$

$$= \int \cos(\omega t) \cos^2(\omega t) \cos(\omega_n t) dt \quad (81)$$

$$= \int \cos\left(\frac{2\pi}{T}t\right) \left[\frac{1}{4} \cos\left(\frac{8\pi}{T}t\right) + \frac{1}{2} \cos\left(\frac{4\pi}{T}t\right) + \frac{1}{4} \right] dt \quad (82)$$

$$= \int \frac{1}{8} \cos\left(\frac{10\pi}{T}t\right) + \frac{1}{8} \cos\left(-\frac{6\pi}{T}t\right) + \frac{1}{4} \cos\left(\frac{6\pi}{T}t\right) \quad (83)$$

$$+ \frac{1}{4} \cos\left(-\frac{2\pi}{T}t\right) + \frac{1}{4} \cos\left(\frac{2\pi}{T}t\right) dt \quad (84)$$

$$= \int \frac{1}{8} \cos\left(\frac{10\pi}{T}t\right) + \frac{3}{8} \cos\left(\frac{6\pi}{T}t\right) + \frac{1}{2} \cos\left(\frac{2\pi}{T}t\right) dt \quad (85)$$

$$= \frac{T}{80\pi} \sin\left(\frac{10\pi}{T}t\right) + \frac{3T}{48\pi} \sin\left(\frac{6\pi}{T}t\right) + \frac{T}{4\pi} \sin\left(\frac{2\pi}{T}t\right) + C \quad (86)$$

References

- [1] Jacob Millman. *Vacuum-tube and Semiconductor Electronics*. McGraw-Hill Book Company, New York.
- [2] Dr. Anthony E. Parker.
- [3] J Seymour. *Electronic devices and components*. Pitman Publishing Limited, 128 Long Acre, london WC2E 9AN, 1981.